MATHEMATICS

A CHARACTERIZATION OF RELATIVELY COMPACT SETS OF HOLOMORPHIC MAPPINGS

BY

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E and F represent complex locally convex spaces, U is open and nonvoid in E; without much loss of generality, we assume that F is separated. Our purpose it to establish a characterization of the relatively compact subsets of the locally convex space $\mathscr{H}(U;F)$ of all holomorphic mappings from U into F, where $\mathscr{H}(U;F)$ is endowed with the Nachbin topology $\mathscr{T}_\omega$. The result is in the Montel style that bounded subsets are identical to relatively compact subsets, when F itself has that Montel property. It extends to locally convex spaces a result known for normed spaces [3], [4]. We also indicate some instances in which the extended result applies beyond the normed case.

We shall follow the terminology of [5]. Let us recall briefly some needed facts. A collection $\mathscr{X}$ of mappings from U into F is amply bounded if, for any continuous seminorm β on F, the collection $\beta \circ \mathscr{X}$ is locally bounded; see [5], § 1.

We shall consider the vector space $\mathscr{P}(^mE;F)$ of all continuous m-homogeneous polynomials from E into F, where $m \in \mathbf{N}$, endowed with the limit topology $\mathscr{T}_\lambda$ whose definition we recall; see [5], § 7. Let α and β denote continuous seminorms on E and F, and E_α and F_β represent E and F seminormed by α and β, respectively. Then $\mathscr{P}(^mE_\alpha;F_\beta)$ is seminormed accordingly. We have

$$\mathscr{P}(^mE;F_\beta)=\bigcup_\alpha \mathscr{P}(^mE_\alpha;F_\beta)$$

and endow $\mathscr{P}(^mE;F_\beta)$ with the corresponding inductive limit topology We also have

$$\mathscr{P}(^mE;F)=\bigcap_\beta \mathscr{P}(^mE;F_\beta)$$

and endow $\mathscr{P}(^mE;F)$ with the corresponding projective limit topology $\mathscr{T}_\lambda$.

We recall the definition of the Nachbin topology $\mathscr{T}_\omega$ on $\mathscr{H}(U;F)$ and refer to [1] for details. A seminorm p on $\mathscr{H}(U;F)$ is ported by a compact

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subset K of U if there is a continuous seminorm β on F for which to every open subset V of U containing K there corresponds $c(V)>0$ such that

$$p(f)\leq c(V)\cdot \sup_{x\in V}\beta[f(x)]$$

for every $f\in\mathscr{H}(U;F)$. Then $\mathscr{T}_\omega$ is defined by all seminorms on $\mathscr{H}(U;F)$ each of which is ported by some compact subset of U. An amply bounded subset of $\mathscr{H}(U;F)$ is bounded for $\mathscr{T}_\omega$.

We need also the topology $\mathscr{T}_{\infty\lambda}$ on $\mathscr{H}(U;F)$ that is defined by the family of seminorms

$$f\in\mathscr{H}(U;F)\mapsto \sup_{x\in K}\tau[\hat{d}^m f(x)]\in\mathbf{R}$$

where $m\in\mathbf{N}$, and K is a compact subset of U, and τ is a continuous seminorm on $\mathscr{P}(^mE;F)$ for $\mathscr{T}_\lambda$.

Proposition. Assume that each bounded subset of $\mathscr{H}(U;F)$ for $\mathscr{T}_\omega$ is amply bounded; and that $\mathscr{T}_\omega$ and $\mathscr{T}_{\infty\lambda}$ induce the same topology on each amply bounded subset of $\mathscr{H}(U;F)$. In order that a subset $\mathscr{X}$ of $\mathscr{H}(U;F)$ be relatively compact for $\mathscr{T}_\omega$ it is necessary and sufficient that:

1) $\mathscr{X}$ be bounded for $\mathscr{T}_\omega$.
2) For every $m\in\mathbf{N}$ and every $x\in U$, the subset

$$\hat{d}^m\mathscr{X}(x)=\{\hat{d}^m f(x);\, f\in\mathscr{X}\}$$

of $\mathscr{P}(^mE;F)$ be relatively compact for $\mathscr{T}_\lambda$.

Proof. The conditions 1) and 2) are necessary. $\mathscr{X}$ being a subset of $\mathscr{H}(U;F)$ that is relatively compact for $\mathscr{T}_\omega$ then $\mathscr{X}$ is bounded for $\mathscr{T}_\omega$. For every $m\in\mathbf{N}$ and every $x\in U$, the definition of $\mathscr{T}_{\infty\lambda}$ on $\mathscr{H}(U;F)$ implies that the mapping

$$f\in\mathscr{H}(U;F)\mapsto \hat{d}^m f(x)\in\mathscr{P}(^mE;F)$$

is continuous from $\mathscr{T}_{\infty\lambda}$ into $\mathscr{T}_\lambda$. Therefore it remains continuous from $\mathscr{T}_\omega$ into $\mathscr{T}_\lambda$ because $\mathscr{T}_{\infty\lambda}\subset\mathscr{T}_\omega$. Since $\mathscr{X}$ is relatively compact in $\mathscr{H}(U;F)$ for $\mathscr{T}_\omega$ then $\hat{d}^m\mathscr{X}(x)$ is relatively compact in $\mathscr{P}(^mE;F)$ for $\mathscr{T}_\lambda$.

The conditions 1) and 2) are sufficient. Let $\mathscr{X}$ satisfy them. Call $\mathscr{T}_p$ the pointwise topology on $\mathscr{H}(U;F)$. Then the closure of $\mathscr{X}$ for $\mathscr{T}_p$ is amply bounded, because $\mathscr{X}$ itself is amply bounded. Since $\mathscr{T}_p\subset\mathscr{T}_{\infty\lambda}\subset\mathscr{T}_\omega$, we have $\mathrm{cl}\,(\mathscr{X},\mathscr{T}_\omega)\subset\mathrm{cl}\,(\mathscr{X},\mathscr{T}_{\infty\lambda})\subset\mathrm{cl}\,(\mathscr{X},\mathscr{T}_p)$ for the corresponding closures. By assumption, $\mathscr{T}_\omega$ and $\mathscr{T}_{\infty\lambda}$ induce the same topology on every amply bounded subset of $\mathscr{H}(U;F)$ and, in particular, on $\mathrm{cl}\,(\mathscr{X},\mathscr{T}_p)$. It follows that $\mathrm{cl}\,(\mathscr{X},\mathscr{T}_\omega)=\mathrm{cl}\,(\mathscr{X},\mathscr{T}_{\infty\lambda})$ and, moreover, that $\mathscr{X}$ is relatively compact in $\mathscr{H}(U;F)$ for $\mathscr{T}_\omega$ if (and only if) the same is true for $\mathscr{T}_{\infty\lambda}$.

To establish this point, let us consider

$$S = \prod_{m=0}^{\infty} \mathscr{P}(^mE; F)$$

that is, the vector space of all formal continuous power series from E into F, endowed with the cartesian product topology when we put on each $\mathscr{P}(^mE; F)$ its topology $\mathscr{T}_\lambda$ ($m \in \mathbf{N}$). Letting $\mathscr{C}(U; S)$ be the vector space of all continuous mappings from U into S, we define

$$\Phi: \mathscr{H}(U; F) \to \mathscr{C}(U; S)$$

in the following way: if $f \in \mathscr{H}(U; F)$ then

$$\Phi(f)(x) = (\hat{d}^m f(x))_{m \in \mathbf{N}} \in S.$$

We have $\Phi(f) \in \mathscr{C}(U; S)$ indeed because its components are the mappings

$$\hat{d}^m f \in \mathscr{H}(U; \mathscr{P}(^mE; F))$$

which are therefore continuous from U into $\mathscr{P}(^mE; F)$ for every $m \in \mathbf{N}$. It is clear that Φ is injective. We can also verify that Φ is a homeomorphism when $\mathscr{H}(U; F)$ is endowed with $\mathscr{T}_{\infty\lambda}$ and $\mathscr{C}(U; S)$ has the compact-open topology. It is also true that the image $\Phi[\mathscr{H}(U; F)]$ is closed in $\mathscr{C}(U; S)$; to avoid misunderstanding at this point, we emphasize that we do not have to assume here that F is differentially stable (see § 4 of [5]). In order to prove that $\mathscr{X}$ is relatively compact in $\mathscr{H}(U; F)$ for $\mathscr{T}_{\infty\lambda}$ it is just the same to show that $\Phi(\mathscr{X})$ is relatively compact in $\mathscr{C}(U; S)$ for the compact-open topology. To apply Ascoli's theorem, it is enough to establish that:

(a) $\Phi(\mathscr{X})(x)$ is relatively compact in S for any $x \in U$.
(b) $\Phi(\mathscr{X})$ is equicontinuous.

Now, condition (a) is equivalent to saying that, for every $x \in U$, the subset

$$\{(\hat{d}^m f(x))_{m \in \mathbf{N}};\, f \in \mathscr{X}\}$$

is relatively compact in S; but this follows from condition 2) in the Proposition and from the compactness criterion in the cartesian product space S.

Condition (b) is equivalent to proving that, for every $m \in \mathbf{N}$, the subset

$$\hat{d}^m \mathscr{X} = \{\hat{d}^m f;\, f \in \mathscr{X}\}$$

of $\mathscr{C}(U; \mathscr{P}(^mE; F))$ is equicontinuous. Since $\mathscr{X}$ is bounded for $\mathscr{T}_\omega$, hence amply bounded, then the Cauchy inequality implies that $\hat{d}^m \mathscr{X}$ is amply bounded too, hence equicontinuous.

This completes the proof. QED

In order to indicate examples within the range of application of the above proposition, let us recall the following result (see Theorem 3.1 of [1]):

LEMMA. Let F be seminormed. Consider the set Γ of all continuous seminorms α on E such that, for every $m \in \mathbb{N}$, the topology $\mathscr{T}_\lambda$ on $\mathscr{P}(^mE; F)$ induces on $\mathscr{P}(^mE_\alpha; F)$ the topology defined by α and the given seminorm on F. Assume that Γ is directed, and that the topology defined on E by Γ coincides with the given topology on E. Then $\mathscr{T}_\omega$ and $\mathscr{T}_{\infty\lambda}$ induce the same topology on every locally bounded subset of $\mathscr{H}(U; F)$.

EXAMPLE 1. Assume that

$$E = \prod_{i \in I} E_i$$

where each E_i is a seminormed space. It is known (see Proposition 2 of [2]) that, for every F, any bounded subset of $\mathscr{H}(U; F)$ for the compact-open topology $\mathscr{T}_0$ is amply bounded; in other words, E is holomorphically infrabarreled (see § 10 of [5]). Since $\mathscr{T}_0 \subset \mathscr{T}_\omega$ it follows that condition 1) of the Proposition is satisfied. It results from the above Lemma that condition 2) of the Proposition is also satisfied.

EXAMPLE 2. If the topology of E is the weak topology $\sigma(E, E')$ it is known (see Proposition 1 of [2]) that, for every F, condition 1) of the Proposition is satisfied; notice that E is not necessarily holomorphically infrabarreled (see § 10 of [5]). It results from the above Lemma that condition 2) of the Proposition is also satisfied.

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